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Parametric instabilities in randomly inhomogeneous plasma

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Abstract. Nonlinear parametric wave interaction in a randomly inhomogeneous plasma is investigated. One effective growth rate and threshold for the mean amplitude and intensity are calculated from Bourret's integral equation. It is found that in a statistically homogeneous medium the effect of fluctuation is to enhance the threshold values but to make the growth weaker.

1. Introduction

Extensive work has been done on the interaction of intense electromagnetic waves with homogeneous and inhomogeneous plasma (Coste *et al* 1975, Porkolab 1978). In all these works the plasma is assumed to be tranquil. However, the plasma produced by irradiating pellets with a laser or some other agency promotes some fluctuations and is likely to be turbulent in general. The background turbulence will create random density fluctuation and may influence the propagation characteristics of the interacting waves of the parametric coupling process. The investigation of parametric instability in presence of background turbulence in a plasma is of great relevance to laser fusion problems.

In this paper the effect of quasistatic random density fluctuation on the scattering of an intense electromagnetic wave off an electrostatic plasma wave is discussed. The assumption that the density fluctuation is quasistatic is justified if the time period of the density fluctuation is long compared with the time of growth of instability of the waves. We assume that the fluctuation is time independent and the degree of random inhomogeneity is small. One can derive an equation for the ensemble average of the wave amplitude $\langle E \rangle$ with its phase as a random function of position in the framework of Bourret's integral equation, discussed in Van Kampen (1976).

2. Method of calculation

One model (Vekshtein and Zaslavskii 1967) that is often assumed for laser interaction with random turbulent plasma is the stochastic variation of light phase. According to this model, the effects of fluctuations can be interpreted as resulting from interference between waves which have acquired different phase shifts during their passage through random inhomogeneities. Valco and Oberman (1973) considered the problem where the phase changes with many small random jumps and diffuses with a diffusion

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coefficient which is shown to be equal to the band width. Tarmor (1973) applied the model of the Poisson distribution of jumps first introduced by Zaslavskii and Zakhavov (1967), with the additional assumption that after the jumps the phase becomes decorrelated. Thomson (1975) solved the purely temporal problem with the Kubo–Anderson process, while Laval *et al* (1977) have extended it to the problem in both space and time. In this paper we shall recall the usual coupled mode equations with a stochastic driver whose phase only stochastically changes as a Uhlenbeck–Ornstein process. We shall discuss its importance in the next section.

3. The coupled mode equation

The equations which describe the slow varying evolution of complex amplitudes in a plasma medium with random optical index can be written as

$$\left(\frac{\partial}{\partial t}+V_1\frac{\partial}{\partial x}+\nu_1\right)a_1=\gamma_0\eta a_2, \qquad \left(\frac{\partial}{\partial t}+V_2\frac{\partial}{\partial x}+\nu_2\right)a_2=\gamma_0\eta^* a_1, \tag{1}$$

where $\eta(x)=\exp(\mathrm{i}\int_0^x K(y)\,\mathrm{d}y)$ and the phase $\phi(x)=\int_0^x K(y)\,\mathrm{d}y$ changes randomly. We ignore its time dependence. a_1 and a_2 are the complex amplitudes: V_1 and V_2, ν_1 and ν_2 are the group velocities and damping rates of the decay waves respectively. In contrast to the Kubo–Anderson process where $\phi(x, t)=S(t-x/V_0)$, we assume the static approximation in our problem, i.e. the Uhlenbeck–Ornstein process as recommended by Brissand and Frisch (1974). Hence the process of ϕ changes is Gaussian and its correlation time is infinitely short ($\tau_c\to 0$) and Markovian. Doob's theorem (1942) asserts that the only Gaussian stationary Markov process is the Uhlenbeck–Ornstein process. It has the property of zero mean value and the autocorrelation function (Uhlenbeck and Ornstein 1930)

$$\left\langle \exp\left(\mathrm{i}\int_0^x K(y_1)\,\mathrm{d}y_1\right)\exp\left(-\mathrm{i}\int_0^{x'} K(y_2)\,\mathrm{d}y_2\right)\right\rangle$$
$$=\exp(\sigma^2/\beta^2)[1-\exp(-\beta|x-x'|)-\beta|x-x'|]$$

where the generalised Reynold's number K in this case will have the form $K=\sigma L$, $L=1/\beta$ is the correlation length and $\sigma^2=\langle\phi^2(0)\rangle$ is the covariance.

The main advantage of taking the Uhlenbeck–Ornstein process in the present case is that in this process the Bourret approximation turns out to be exact. It will provide a check for different approximations for different generalised Reynold's numbers, as has been discussed by Brissand and Frisch (1974).

3. The Bourret equations for averaged amplitudes and average amplitudes thresholds

Equation (1) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}a_1\\a_2\end{pmatrix}=A\begin{pmatrix}a_1\\a_2\end{pmatrix} \tag{2}$$

with

$$A = A_0 + \gamma_0 A_1,$$

$$A_0 = \begin{pmatrix} -(\mathrm{i}\omega + \nu_1)/V_1 & 0 \\ 0 & -(\mathrm{i}\omega + \nu_2)/V_2 \end{pmatrix}, \tag{3}$$

$$A_1 = \begin{pmatrix} 0 & \eta/V_1 \\ \eta^*/V_2 & 0 \end{pmatrix}. \tag{4}$$

Let us now write the Bourret integral equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle U(x)\rangle = A_0\langle U(x)\rangle + \gamma_0^2 \int_0^x \langle A_1(x)\exp[A_0(x-x')A_1(x')]\rangle\langle U(x')\rangle\,\mathrm{d}x$$

from which follows a differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle U(x)\rangle = A_0\langle U(x)\rangle + \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\langle U(x)\rangle \tag{5}$$

where one has $\langle A_1(x)A_1(x')\rangle \approx 0$ when $x - x' > x_c$. Hence as soon as $x > x_c$ no error is made by extending the integral to infinity, where

$$P_1 = \int_0^\infty \frac{\gamma_0^2}{V_1 V_2}\exp\left[-\left(\frac{\mathrm{i}\omega + \nu_2}{V_2}\right)(x-x')\right]$$
$$\times\left\langle \exp\left(\mathrm{i}\int_0^x K(y_1)\,\mathrm{d}y_1\right)\exp\left(-\mathrm{i}\int_0^{x'} K(y_2)\,\mathrm{d}y_2\right)\right\rangle \mathrm{d}x', \tag{6}$$

$$P_2 = \int_0^\infty \frac{\gamma_0^2}{V_1 V_2}\exp\left[-\left(\frac{\mathrm{i}\omega + \nu_1}{V_1}\right)(x-x')\right]$$
$$\times\left\langle \exp\left(-\mathrm{i}\int_0^x K(y_1)\,\mathrm{d}y_1\right)\exp\left(\mathrm{i}\int_0^{x'} K(y_2)\,\mathrm{d}y_2\right)\right\rangle \mathrm{d}x'. \tag{7}$$

with the autocorrelation function mentioned before,

$$\left\langle \exp\left(\mathrm{i}\int_0^x K(y_1)\,\mathrm{d}y_1\right)\exp\left(-\mathrm{i}\int_0^{x'} K(y_2)\,\mathrm{d}y_2\right)\right\rangle$$
$$= \exp(\sigma^2/\beta^2)[1 - \exp(-\beta|x-x'|) - \beta|x-x'|]. \tag{8}$$

Substituting equation (8) in equation (6), one obtains

$$P_1 = \frac{\gamma_0^2}{V_1 V_2}\exp\left(\frac{\sigma^2}{\beta^2}\right)\int_0^\alpha \exp\left(-nx - \frac{\sigma^2}{\beta^2}[\exp(-\beta x) + \beta x]\right)\mathrm{d}x \tag{9}$$

with $\eta = (\mathrm{i}\omega + \nu_2)/V_2$. Then

$$P_1 = \frac{\gamma_0^2}{V_1 V_2}\exp\left(\frac{\sigma^2}{\beta^2}\right)\frac{1}{\beta}\left(\frac{\beta^2}{\sigma^2}\right)^{-n/\beta}\gamma\left(a, \frac{\sigma}{\beta^2}\right) \tag{10}$$

where $a = n/\beta + \sigma^2/\beta^2$ and $\gamma(a, x)$ is the incomplete Gamma function defined by (Abramowitz and Stegun 1965)

$$\gamma(a, x) = \int_0^x \exp(-t)t^{a-1}\,\mathrm{d}t.$$

Now the mean amplitudes can be written in the following form:

$$\langle a_1\rangle = \exp\left(-\frac{(\mathrm{i}\omega+\nu_1)}{V_1}x + P_1 x\right), \tag{11}$$

$$\langle a_2\rangle = \exp\left(-\frac{(\mathrm{i}\omega+\nu_2)}{V_2}x + P_2 x\right), \tag{12}$$

where P_1 is given by equation (10) and P_2 is obtained from P_1 by replacing ν_2 and V_2 by ν_1 and V_1 respectively. For σ^2/β^2 small, one can expand $\gamma(a, \sigma^2/\beta^2)$ in terms of σ^2/β^2,

$$\gamma(a, x) = x^a \exp(-x) \sum_{m=0}^{\infty} \frac{x_m}{\Gamma(a+m+1)}. \tag{13}$$

Then we would obtain to first order

$$P_1 = \frac{\gamma_0^2}{V_1 V_2} \frac{1}{(\mathrm{i}\omega+\nu^2)/V_2 + \Delta K_0)]} \qquad \text{where } \Delta K_0 = \sigma^2/\beta. \tag{14}$$

This result is identical to that obtained in the Kubo–Anderson process. We obtain a similar result for P_2.

Hence the threshold for $\langle a_1\rangle$ would read (for $V_1 V_2 > 0$)

$$\gamma_0^2 = \nu_1(\nu_2 + V_2 \Delta K_0), \tag{15}$$

and similarly for

$$\gamma_0^2 = \nu_2(\nu_1 + V_1 \Delta K_0). \tag{16}$$

These results agree with those obtained by Laval *et al* (1977), for the case $V_0 = 0$. In particular the coherent threshold is

$$\gamma_0^2 = \nu_1 \nu_2. \tag{17}$$

In the present case the growth rate is given by a transcendental equation (as is evident from equation (10)). However, we can take the lowest-order terms in equation (10), which greatly simplifies matters. Integrals in equation (9) will exist irrespective of the signs of V_1 and V_2. Thus for $|\sigma/\beta| \ll 1$

$$P_1 = \frac{\gamma_0^2}{V_1 V_2} \frac{1}{\beta a} \left[\exp\left(\frac{\sigma^2}{\beta^2}\right)\right] \left(\frac{\sigma^2}{\beta^2}\right)^{\sigma^2/\beta^2}. \tag{18}$$

The threshold for $\langle a_1\rangle$ is given by

$$\gamma_0^2 = \nu_1(\nu_2 + V_2 \Delta K_0) A \tag{19}$$

where

$$A = [\exp(\sigma^2/\beta^2)(\sigma^2/\beta^2)^{\sigma^2/\beta^2}]^{-1}. \tag{20}$$

It is clear that the threshold given by the Uhlenbeck–Ornstein process of equation (19) is greater than that of the Kubo–Anderson process given by equation (15). The integration in equation (9) can be evaluated in terms of the error function. After some straightforward calculations (in the limit $z \to \infty$) (using $\operatorname{erf} z \simeq (1 - 1/2z^2)$), the instability threshold conditions can be written as (for $V_1 V_2 < 0$)

$$\gamma_0^2 = \nu_1 \nu_2 (2\sigma^2) \min(V_1^2/\nu_1^2, V_2^2/\nu_2^2) \tag{21}$$

provided

$$(\sqrt{2}\sigma) > \max(|V_1|/\nu_1, |V_2|/\nu_2). \tag{22}$$

For $\sigma^2 L^2 \ll 1$, the lossless modified growth rate is obtained from equation (14),

$$\Lambda = \gamma_0^2/|V_1 V_2|(\Delta K_0)$$

with

$$\Delta K_0 = |\sigma^2 L| = |\langle(\delta K^2)\rangle L|.$$

This agrees with Bondeson (1977). It may be interpreted as the effect of turbulence which introduces random mismatch in the three interacting waves. In spite of the complicated form of P_1 in equation (10) we obtain the amplification as a function of the normalised growth rate $\gamma_0^2/\nu_1\nu_2$ given in figure 1. Well above the damping threshold this yields

$$\Lambda/\Lambda_0 \simeq (\nu_1/V_1)\{[a' + xQ(2x|2a') - x - a'Q(2x|2a')](\exp a')(a')^{(a'-x)}\Gamma(a) - \Lambda_0\}$$

where $Q(\chi^2|\nu)$ is the chi-square distribution ($\chi^2 = 2x$, $\nu = 2a'$) and $\Gamma(a)$ is the complete gamma function,

$$\Lambda_0 = \gamma_0^2/\nu_1\nu_2, \qquad \gamma(a', x) = \Gamma(a')[1 - Q(2x|2a')],$$

$$a' = \nu_2 L/V_2 + \sigma^2 L^2, \qquad x = \sigma^2 L^2.$$

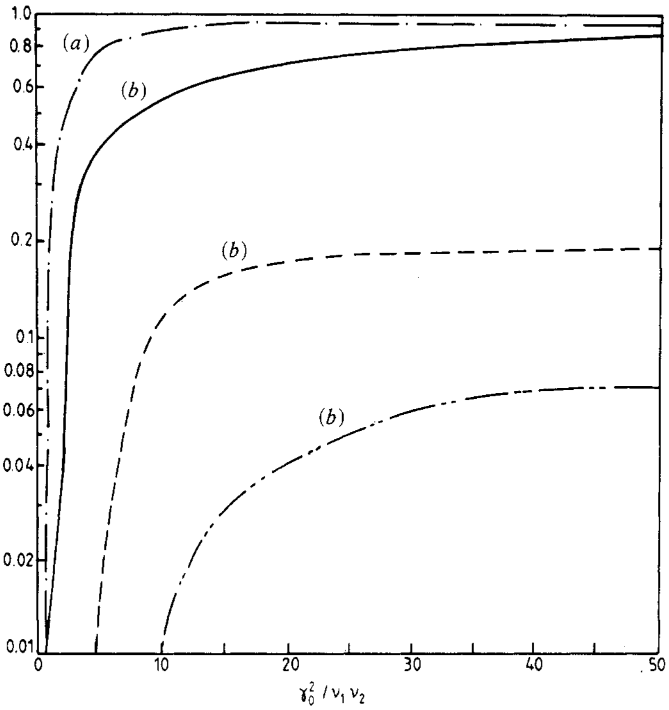

Figure 1. (*a*) The growth rate in the statistically homogeneous plasma is compared with work of others (Anderson 1977). (*b*) The growth rates in a weakly random inhomogeneous plasma are shown. —·—· $\sigma^2L^2 = 0$, $a' = 1$; —— $\sigma^2L^2 = 0$, $a' = 1$; - - - $\sigma^2L^2 = 0.25$, $a' = 0.8$; —··—·· $\sigma^2L^2 = 0.45$, $a^1 = 0.5$.

Assuming $\sigma L=0$, this reduces (from equation (15)) to

$$\Lambda/\Lambda_0=(\nu_1/V_1)(1-\nu_1\nu_2/\gamma_0^2).$$

This is compared with the result of Anderson (1977) in figure 1 ($\Lambda/\Lambda_0=1-(4/\pi)(\nu_1\nu_2/\gamma_0^2)^{1/2}$). The curves differ in slope only at the intermediate values of $\gamma_0^2/\nu_1\nu_2$. It is interesting to note that the degree of inhomogeneous fluctuation increases the threshold values but decreases the growth rate.

4. Power stability condition

The Bourret equation for the average intensity can be written in the following way, outlined in § 3:

$$\frac{\partial}{\partial x}\left\langle\begin{pmatrix} a_1^2 \\ a_1a_2 \\ a_2^2\end{pmatrix}\right\rangle = B\left\langle\begin{pmatrix} a_1^2 \\ a_1a_2 \\ a_2^2\end{pmatrix}\right\rangle \tag{23}$$

where we have

$$\beta=\begin{pmatrix} -\frac{(\mathrm{i}\omega+\nu_1)}{V_1}+P_{11} & 0 & P_{13} \\ 0 & -\left[\mathrm{i}\omega\left(\frac{1}{V_2}-\frac{1}{V_1}\right)+\frac{\nu_1}{V_1}+\frac{\nu_2}{V_2}\right]+P_{22} & 0 \\ P_{31} & 0 & -\left(\frac{\mathrm{i}\omega+\nu_2}{V_2}\right)+P_{33}\end{pmatrix} \tag{24}$$

where

$$P_{11}=\frac{\gamma_0^2}{V_1V_2}\int_0^\infty \langle\eta(x)\eta^*(x')\rangle \exp-\left[\mathrm{i}\omega\left(\frac{1}{V_2}-\frac{1}{V_1}\right)+\frac{\nu_1}{V_1}+\frac{\nu_2}{V_2}\right](x-x')\,\mathrm{d}x', \tag{25}$$

$$P_{22}=\frac{\gamma_0^2}{V_1V_2}\int_0^\infty \langle\eta(x)n^*(x')\rangle\left\{\exp\left[-\left(\frac{\mathrm{i}\omega+\nu_1}{V_1}\right)\right](x-x')\right.$$
$$\left.+\exp\left[-\left(\frac{\mathrm{i}\omega+\nu_2}{V_2}\right)\right](x-x')\right\}\mathrm{d}x', \tag{26}$$

$$P_{13}=\frac{\gamma_0^2}{V_1V_2}\int_0^\infty \langle\eta(x)\eta(x')\rangle \exp\left[-\left(\frac{\mathrm{i}\omega+\nu_1}{V_1}+\frac{\mathrm{i}\omega+\nu_2}{V_2}\right]\right](x-x')\,\mathrm{d}x', \tag{27}$$

$$P_{31}=\frac{\gamma_0^2}{V_1V_2}\int_0^\infty \langle\eta^*(x)\eta^*(x')\rangle \exp\left[-\left(\frac{\mathrm{i}\omega+\nu_1}{V_1}+\frac{\mathrm{i}\omega+\nu_2}{V_2}\right)\right](x-x')\,\mathrm{d}x', \tag{28}$$

$P_{11}=P_{33}$,

which after integration becomes

$$P_{11}=\frac{\gamma_0^2}{\nu_1\nu_2}\frac{1}{\beta}\left[\exp\left(\frac{\sigma^2}{\beta^2}\right)\right]\gamma\left(a,\frac{\sigma^2}{\beta^2}\right)\left(\frac{\sigma^2}{\beta^2}\right)^{-\nu_{11}/\beta}, \tag{29}$$

$$P_{22}=\frac{\gamma_0^2}{\nu_1\nu_2}\frac{1}{\beta}\left[\exp\left(\frac{\sigma^2}{\beta^2}\right)\right]\gamma\left(\bar{a},\frac{\sigma^2}{\beta^2}\right)\left(\frac{\sigma^2}{\beta^2}\right)^{-\nu_{22}/\beta}, \tag{30}$$

$$P_{13}=P_{31}=0, \tag{31}$$

with

$$a=\nu_{11}/\beta+\sigma^2/\beta^2, \qquad \bar{a}=\nu_{22}/\beta+\sigma^2/\beta^2, \tag{32}$$

$$\nu_{11}=\left[\mathrm{i}\omega\left(\frac{1}{V_2}-\frac{1}{V_1}\right)+\frac{\nu_1}{V_1}+\frac{\nu_2}{V_2}\right], \tag{33}$$

$$\nu_{22}=\left[\mathrm{i}\omega\left(\frac{1}{V_1}-\frac{1}{V_2}\right)+\frac{\nu_1}{V_1}+\frac{\nu_2}{V_2}\right]. \tag{34}$$

However, the integrands for P_{13} cannot be expressed in terms of $(x-x')$.

In the limit $x\to\infty$, $P_{13}=P_{31}=0$. The final result is

$$\frac{\partial}{\partial x}\left\langle\begin{pmatrix} a_1^2 \\ a_1a_2 \\ a_2^2\end{pmatrix}\right\rangle = D\left\langle\begin{pmatrix} a_1^2 \\ a_1a_2 \\ a_2^2\end{pmatrix}\right\rangle, \tag{35}$$

$$D=\begin{pmatrix} -\dfrac{(\mathrm{i}\omega+\nu_1)}{V_1}+P_{11} & 0 & 0 \\ 0 & -\left[\mathrm{i}\omega\left(\dfrac{1}{V_2}-\dfrac{1}{V_1}\right)+\dfrac{\nu_1}{V_1}+\dfrac{\nu_2}{V_2}\right]+P_{22} & 0 \\ 0 & 0 & -\left(\dfrac{\mathrm{i}\omega+\nu_2}{V_2}\right)+P_{33}\end{pmatrix} \tag{36}$$

As before the threshold condition for small σ^2/β^2 can be expressed in a simple form:

$$\begin{gathered}\nu_1\nu_2\exp(-\sigma^2L^2)=\gamma_0^2[1/\bar{\nu}+\sigma^2L^2/(\bar{\nu}-1)], \\ \bar{\nu}=\nu_1/V_1+\nu_2/V_2.\end{gathered} \tag{37}$$

For $V_1V_2<0$ the absolute instability threshold can be obtained in the manner outlined in § 3, and it reads

$$\gamma_0^2=\nu_1\nu_2\frac{2\sigma^2}{\nu_1/|V_1|+\nu_2/|V_2|}\min\left(\frac{|V_1|}{\nu_1},\frac{|V_2|}{\nu_2}\right) \tag{38}$$

with the validity conditions

$$\sqrt{2}\sigma>\nu_1/|V_1|+\nu_2/|V_2|.$$

From (21) and (38) it follows that to first order, the absolute instability threshold for $\langle a, a^*\rangle$ is lower than that for $\langle a\rangle$, a result obtained by Laval *et al* (1977).

5. Discussion

In this paper the parametric coupling processes have been investigated in presence of background density fluctuations. The results are applicable to stimulated Raman scattering and can readily be extended to stimulated Brillouin scattering and other laser fusion decay instabilities. In figure 1 it is shown that the density fluctuations have enhanced the thresholds but quenched the amplification growth rate.

We have found explicit expressions for mean amplitudes and intensities in a series of partial waves. It was pointed out by Laval *et al* (1977) that in the Kubo–Anderson process, the case $V_1 V_2 > 0$ can be solved exactly. In the present case we have taken a more general case and obtained the growth rate as a transcendental function of the frequencies and velocities. Hence it is not possible to compare our exact results with those of Laval *et al* (1977). However, to first order the conclusion drawn from our work agrees with their results.

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